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Approximation by the Solutions of the Heat Equation

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Solutions of the heat equations

$$\begin{cases} u_{xx}(x,t) = u_t(x,t) & \text{on } (-\infty,\infty) \times \{t > 0\} \\ u(x,0) = F(x) & \text{on } (-\infty,\infty), \end{cases}$$

where F belongs to the weighted $L^2[\mathbb{R}, \exp(-ax^2)\,dx]$ space, are used in order to study best approximation problems.

1. INTRODUCTION

Throughout this paper, a is a non-negative constant. Let $\mathbb{R} = (-\infty, \infty)$ and for $1 - 2at > 0$ let

$$W_{a,t}(x) = \frac{-ax^2}{1 - 2at}, \qquad x \in \mathbb{R}.$$

Let $u_F(x,t)$, $0 < t < 1/2a$, denote the solution of the Cauchy problem

$$u_{xx}(x,t) = u_t(x,t) \qquad \text{on } \mathbb{R} \times \{t > 0\} \tag{1.1}$$

and

$$u(x,0) = F(x) \qquad \text{on } \mathbb{R}, \tag{1.2}$$

where $F \in L^2[\mathbb{R}, \exp(-ax^2)\,dx]$. We then have

$$u_F(x,t) \in L^2[\mathbb{R}, \exp\{W_{a,t}(x)\}dx]$$

(see Theorem 2 below).

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Given $h \in L^2[\mathbb{R}, \exp\{W_{a,t}(x)\}\,dx]$, with t fixed, we want to minimize the expression

$$\int_{\mathbb{R}} |u_F(x,t) - h(x)|^2 \exp\{W_{a,t}(x)\}\,dx,$$

where F runs over $L^2[\mathbb{R}, \exp(-ax^2)\,dx]$. We have two main results.

First, we determine a necessary and sufficient condition on h for which there exists $F \in L^2[\mathbb{R}, e^{-ax^2}\,dx]$ with

$$\int_{\mathbb{R}} |u_F(x,t) - h(x)|^2 \exp\{W_{a,t}(x)\}\,dx = 0.$$

Since if $u_F(x,t)$ is a solution of (1.1) and (1.2) then $u_F(z,t)$ is an entire function of z, the condition on h is connected with the analytic extension of h to the complex plane.

Secondly, we will construct for any given h in $L^2[\mathbb{R}, \exp\{W_{a,t}(x)\}\,dx]$ the minimizing sequence $\{u_{F_n}\}$ for which

$$\lim_{n\to\infty} \int_{\mathbb{R}} |u_{F_n}(x,t) - h(x)|^2 \exp\{W_{a,t}(x)\}\,dx = 0.$$

Some aspects of our method have appeared in [1].

2. Solution Space

We first characterize the Hilbert space formed by the solutions $u_F(x,t)$ of (1.1) and (1.2) using the theory of reproducing kernels [5].

We define a linear operator L on functions $F \in L^2[\mathbb{R}, e^{-ax^2}\,dx]$ by

$$\begin{aligned} LF(x) &= \left(F(\xi), h(x-\xi,t)e^{a\xi^2}\right)_{L^2[\mathbb{R}, e^{-a\xi^2}d\xi]} \\ &= u_F(x,t), \end{aligned}$$

where $h(x,t)$ is the heat kernel $(1/\sqrt{4\pi t})e^{-x^2/4t}$. Then the range of L forms a Hilbert space $H_{K(a,t)}$ with the reproducing kernel

$$\begin{aligned} K(z,\bar{u};a,t) &= \left(h(z-\xi,t)e^{a\xi^2}, h(u-\xi,t)e^{a\xi^2}\right)_{L^2[\mathbb{R}, e^{-a\xi^2}d\xi]} \\ &= \int_{\mathbb{R}} h(z-\xi,t)h(\bar{u}-\xi,t)e^{a\xi^2}\,d\xi \\ &= \frac{1}{2\sqrt{2\pi t(1-2at)}} \exp\left\{-\frac{1-4at}{8t(1-2at)}z^2\right\} \\ &\quad \cdot \exp\left\{-\frac{1-4at}{8t(1-2at)}\bar{u}^2\right\} \exp\left\{\frac{z\bar{u}}{4t(1-2at)}\right\}. \end{aligned} \tag{2.1}$$

Furthermore, the norm in $H_{K(a,t)}$ satisfies the isometrical identity

$$\|u_F(\cdot,t)\|_{H_{K(a,t)}} = \|F\|_{L^2[\mathbb{R}, e^{-a\xi^2} d\xi]}.$$

By the arguments in [7, Theorems 3.1 and 4.1] we obtain

THEOREM 1. *For any fixed t, $1-2at>0$, the solutions $u_F(x,t)$ of (1.1)–(1.2) are extensible analytically on $\mathbb{C}$ in the form $u_F(z,t)$ and form the reproducing kernel Hilbert space $H_{K(a,t)}$ with the norm*

$$\|u_F(\cdot,t)\|^2_{H_{K(a,t)}} = \frac{1}{\sqrt{2\pi t(1-2at)}} \iint_{\mathbb{C}} |u_F(z,t)|^2 \cdot \exp\left\{ W_{a,t}(x) - \frac{y^2}{2t} \right\} dx\, dy.$$

Furthermore the initial functions F are given by

$$F(\xi) = s - \lim_{N\to\infty} \frac{e^{\alpha\xi^2}}{\sqrt{2\pi t(1-2at)}} \iint_{|z|<N} u_F(z,t)\overline{h(z-\xi,t)} \cdot \exp\left\{ W_{a,t}(x) - \frac{y^2}{2t} \right\} dx\, dy,$$

where the limit is taken on $L^2(\mathbb{R}, \exp(-a\xi^2)\, d\xi]$.

Note that

$$(u_F(\cdot,t), u_G(\cdot,t))_{H_{K(a,t)}} = (F,G)_{L^2[\mathbb{R}, e^{-a\xi^2} d\xi]}.$$

Thus the second part of Theorem 1 follows formally from

$$\begin{aligned} F(\varphi)e^{-a\varphi^2} &= (F, \delta_\varphi)_{L^2[\mathbb{R}, e^{-a\xi^2} d\xi]} \\ &= \left(u_F(\cdot,t), u_{\delta_\varphi}(\cdot,t)\right)_{H_{K(a,t)}} \\ &= (u_F(\cdot,t), h(\cdot - \varphi, t))H_{K(a,t)}, \end{aligned}$$

where δ_φ is the dirac delta centered at φ.

In Theorem 1, from the identity

$$\int_{\mathbb{R}}\int_{|y|<N} u_F(z,t) h(\bar{z}-\varphi,t) \exp\left\{W_{a,t}(x) - \frac{y^2}{2t}\right\} dx\, dy$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} F(x') \exp\{W_{a,t}(x)\}$$

$$\times \left(\int_{|y|<N} h(z-x',t) h(\bar{z}-\varphi,t) e^{-y^2/2t} dy\right) dx'\, dx$$

$$= 2\int_{\mathbb{R}}\left(\int_{\mathbb{R}} F(x') h(x-x',t) \frac{\sin N((x'-\varphi)/2t)}{(x'-\varphi)/2t} dx'\right) h(x-\varphi,t)$$

$$\cdot \exp\{W_{a,t}(x)\}\, dx,$$

we have the pointwise convergence in the form

$$F(\xi) = \lim_{N\to\infty} \frac{e^{a\xi^2}}{\sqrt{2\pi t(1-2at)}} \int_{\mathbb{R}}\int_{|y|<N} u_F(z,t) \overline{h(z-\xi,t)}$$

$$\cdot \exp\left\{W_{a,t}(x) - \frac{y^2}{2t}\right\} dx\, dy.$$

3. Inequality of Fejér–Riesz Type

We define the linear operator $T: H_{K(a,t)} \to L^2[\mathbb{R}, \exp\{W_{a,t}(x)\}\, dx]$ by the restriction Tf of f to $\mathbb{R}$. Then, the following inequality of Fejér–Riesz type (cf. [3]) implies that T is well-defined and bounded on the space $H_{K(a,t)}$.

Theorem 2. *For any $f \in H_{K(a,t)}$, the following inequality holds*:

$$\int_{\mathbb{R}} |f(x)|^2 \exp\{W_{a,t}(x)\}\, dx \le \|f\|^2_{H_{K(a,t)}}. \tag{3.1}$$

Here, the constant 1 *as the coefficient of $\|f\|^2_{H_{K(a,t)}}$ is best possible in the inequality.*

Proof. From the kernel form (2.1) and [2, Sect. 8 in Part I], $f(\in H_{K(a,t)})$ is expressible in the form

$$f(z) = \frac{1}{2\sqrt{2\pi t(1-2at)}} \exp\left\{-\frac{1-4at}{8t(1-2at)} z^2\right\} f_1(z), \tag{3.2}$$

where $f_1(z)$ is a member in the reproducing kernel Hilbert space H_1 determined by the positive matrix

$$K'(z,\bar{u};a,t) = \exp\left\{\frac{z\bar{u}}{4t(1-2at)}\right\}.$$

Moreover, we have the isometrical identity

$$\|f\|^2_{H_{K(a,t)}} = \frac{1}{2\sqrt{2\pi t(1-2at)}}\|f_1\|^2_{H_1}. \tag{3.3}$$

Meanwhile, we recall the identity

$$K'(z,\bar{u};a,t) = \sqrt{\frac{2t(1-2at)}{\pi}}\exp\left\{\frac{-z^2}{8t(1-2at)}\right\}$$
$$\cdot\exp\left\{\frac{-\bar{u}^2}{8t(1-2at)}\right\}\int_{\mathbb{R}} e^{z\xi}e^{\bar{u}\xi}\exp\{-2t(1-2at)\xi^2\}\,d\xi.$$

This representation implies that f_1 is expressible in the form

$$f_1(z) = \sqrt{\frac{2t(1-2at)}{\pi}}\exp\left\{\frac{-z^2}{8t(1-2at)}\right\}$$
$$\cdot\int_{\mathbb{R}} F_1(\xi)e^{z\xi}\exp\{-2t(1-2at)\xi^2\}\,d\xi \tag{3.4}$$

for a function F_1 satisfying

$$\int_{\mathbb{R}}|F_1(\xi)|^2\exp\{-2t(1-2at)\xi^2\}\,d\xi < \infty$$

and we have the identity

$$\|f_1\|^2_{H_1} = \sqrt{\frac{2t(1-2at)}{\pi}}\int_{\mathbb{R}}|F_1(\xi)|^2\exp\{-2t(1-2at)\xi^2\}\,d\xi \tag{3.5}$$

(see [5]). By applying Plancherel's theorem to (3.4), we have, from (3.5)

$$\int_{\mathbb{R}}|f_1(iy)|^2\exp\left\{\frac{-y^2}{4t(1-2at)}\right\}dy$$
$$= 4t(1-2at)\int_{\mathbb{R}}|F_1(\xi)|^2\exp\{-4t(1-2at)\xi^2\}\,d\xi$$
$$\le 4t(1-2at)\int_{\mathbb{R}}|F_1(\xi)|^2\exp\{-2t(1-2at)\xi^2\}\,d\xi$$
$$= 2\sqrt{2\pi t(1-2at)}\,\|f_1\|^2_{H_1}. \tag{3.6}$$

Put $g(z) = f_1(-iz)$. Then, g belongs to H_1, and $\|g\|_{H_1} = \|f_1\|_{H_1}$. Hence, we have

$$\int_{\mathbb{R}} |f_1(x)|^2 \exp\left\{\frac{-x^2}{4t(1-2at)}\right\} dx = \int_{\mathbb{R}} |g(iy)|^2 \exp\left\{\frac{-y^2}{4t(1-2at)}\right\} dy$$
$$\leq 2\sqrt{2\pi t(1-2at)}\, \|f_1\|_{H_1}^2. \qquad (3.7)$$

Hence, from (3.2), (3.3), and (3.7) we have the desired inequality.

Next, we refer to the sharpness of the inequality. For any α: $0 < \alpha < 1$, we set

$$S_\alpha(\xi) = \alpha \exp\{-2t(1-2at)\xi^2\} - \exp\{-4t(1-2at)\xi^2\}.$$

Since the sharpness of the inequality depends on that of the inequality in (3.6), for any α it is sufficient to construct a member in $L^2[\mathbb{R}, \exp\{-2t(1-2at)\xi^2\}\, d\xi]$ satisfying

$$\int_{\mathbb{R}} |G_\alpha(\xi)|^2 S_\alpha(\xi)\, d\xi < 0.$$

However, since $S_\alpha(0) < 0$, the existence of $G_\alpha(\xi)$ is apparent.

4. Existence of Best Approximation

For any fixed $t : 1 - 2at > 0$ and for a function $h \in L^2[\mathbb{R}, \exp\{W_{a,t}(x)\}\, dx]$, we determine a condition for the existence of the approximation $u_F(z,t) \in H_{K(a,t)}$ in the sense that there exists a member F in $L^2[\mathbb{R}, \exp(-ax^2)\, dx]$ such that

$$\int_{\mathbb{R}} |u_F(x,t) - h(x)|^2 \exp\{W_{a,t}(x)\}\, dx = 0. \qquad (4.1)$$

THEOREM 3. *For $h \in L^2[\mathbb{R}, \exp\{W_{a,t}(x)\}\, dx]$, there exists a member $F \in L^2[\mathbb{R}, \exp(-ax^2)\, dx]$ satisfying* (4.1) *if and only if*

$$\iint_{\mathbb{C}} \left| \int_{\mathbb{R}} h(\xi) \exp\left\{-\frac{(1+4at)\xi^2}{8t(1-2at)} + \frac{z\xi}{4t(1-2at)}\right\} d\xi \right|^2$$
$$\cdot \exp\left\{\frac{-3x^2+y^2}{12t(1-2at)}\right\} dx\, dy < \infty. \qquad (4.2)$$

Proof. First, note that the adjoint operator T^* of T is well-defined by Theorem 2, and

$$[T^*G](z) = (G(\cdot), K(\cdot, \bar{z}; a, t))_{L^2[\mathbb{R}, \exp\{W_{a,t}(x)\} dx}$$

for $G \in L^2[\mathbb{R}, \exp\{W_{a,t}(x)\} dx]$. Since $\{K(\cdot, \bar{z}; a, t) | z \in \mathbb{C}\}$ is complete in $L^2[\mathbb{R}, \exp\{W_{a,t}(x)\} dx]$, the operator T^* is one to one. Hence there exists a member F in $L^2[\mathbb{R}, \exp(-ax^2) dx]$ satisfying (4.1) if and only if T^*h belongs to the range of $H_{K(a,t)}$ under the operator T^*T. If f is a member in the range of T^*T, we can write, by a member g in $H_{K(a,t)}$ with $T^*Tg = f$,

$$\begin{aligned} f(z) &= (T^*Tg, K(\cdot, \bar{z}; a, t))_{H_{K(a,t)}} \\ &= (g, T^*TK(\cdot, \bar{z}; a, t))_{H_{K(a,t)}}. \end{aligned} \tag{4.3}$$

Following the general idea in [5], we characterize the members in the range of T^*T. From the expression (4.3), we compute the complex kernel form

$$\begin{aligned} k(z, \bar{u}; a, t) &= (T^*TK(\cdot, \bar{u}; a, t), T^*TK(\cdot, \bar{z}; a, t))_{H_{K(a,t)}} \\ &= (TK(\cdot, \bar{u}; a, t), TT^*TK(\cdot, \bar{z}; a, t))_{L^2[\mathbb{R}, \exp\{W_{a,t}(x)\} dx]}. \end{aligned}$$

Meanwhile,

$$\begin{aligned} [T^*TK(\cdot, \bar{u}; a, t)](z) &= (TK(\cdot, \bar{u}; a, t), TK(\cdot, \bar{z}; a, t))_{L^2[\mathbb{R}, \exp\{W_{a,t}(x)\} dx]} \\ &= \frac{1}{4\sqrt{\pi t(1 - 2at)}} \exp\left\{-\frac{(1 - 8at) z^2}{16t(1 - 2at)}\right\} \\ &\quad \cdot \exp\left\{-\frac{(1 - 8at)\bar{u}^2}{16t(1 - 2at)}\right\} \exp\left\{\frac{z\bar{u}}{8t(1 - 2at)}\right\}. \end{aligned} \tag{4.4}$$

Hence we obtain the identity

$$\begin{aligned} k(z, \bar{u}; a, t) &= \frac{1}{2\sqrt{6\pi t(1 - 2at)}} \exp\left\{-\frac{(1 - 12at) z^2}{24t(1 - 2at)}\right\} \\ &\quad \cdot \exp\left\{-\frac{(1 - 12at)\bar{u}^2}{24t(1 - 2at)}\right\} \exp\left\{\frac{z\bar{u}}{12t(1 - 2at)}\right\}. \end{aligned} \tag{4.5}$$

Meanwhile, from this expression (4.5), we see that $k(z, \bar{u}; a, t)$ is the reproducing kernel for the Hilbert space $H_{k(a,t)}$ consisting of all entire functions g with finite norms

$$\|g\|^2_{H_{k(a,t)}} = \frac{1}{\sqrt{6\pi t(1-2at)}} \iint_{\mathbb{C}} |g(z)|^2 \cdot \exp\left\{ W_{a,t}(x) + \frac{(6at-1)y^2}{6t(1-2at)} \right\} dx\,dy \tag{4.6}$$

(see the similar argument in [6]). From the representation of T^*, we have the desired theorem.

5. Representation of Analytic Extension

For h in $L^2[\mathbb{R}, \exp\{W_{a,t}(x)\}\,dx]$ satisfying (4.2), note that from (4.1), $h(x)$ is extensible analytically onto $\mathbb{C}$ except for a set of measure zero and the analytic extension $h(z)$ belongs to $H_{K(a,t)}$. We give the following representation of $h(z)$ in terms of $h(\xi)$:

Theorem 4. *For a member in* $L^2[\mathbb{R}, \exp\{W_{a,t}(x)\}\,dx]$ *satisfying* (4.2), *the analytic extension* $h(z)$ *is expressible in the form*

$$h(z) = \frac{1}{16\sqrt{3}\,\{\pi t(1-2at)\}^{3/2}} \exp\left\{ -\frac{(1-8at)z^2}{16t(1-2at)} \right\} \cdot \iint_{\mathbb{C}} \left[\int_{\mathbb{R}} h(\xi) \exp\left\{ -\frac{(1+4at)\xi^2}{8t(1-2at)} + \frac{Z\xi}{4t(1-2at)} \right\} d\xi \right] \cdot \exp\left\{ \frac{(19-16at)X^2}{16t(1-2at)} + \frac{(-65+48at)Y^2}{48t(1-2at)} + \frac{\bar{Z}z + iXY}{8t(1-2at)} \right\} dX\,dY,$$

$$Z = X + iY. \tag{5.1}$$

Proof. Let S be the adjoint operator of T^*T from $H_{k(a,t)}$ to $H_{K(a,t)}$. Then, since T^*T is an isometry of $H_{K(a,t)}$ onto $H_{k(a,t)}$, the operator S is the inverse of T^*T. Moreover, $T^*h = T^*TST^*h$, and $h = TST^*h$. Hence, from (4.6) we have the desired representation

$$\begin{aligned} h(z) &= [ST^*h](z) \\ &= (ST^*h, K(\cdot, \bar{z}; a, t))_{H_{K(a,t)}} \\ &= (T^*h, T^*TK(\cdot, \bar{z}; a, t))_{H_{k(a,t)}}. \end{aligned}$$

Meanwhile, from Lemma 1 in [4] we can obtain another representation of $h(z)$.

Since $h(z)$ belongs to the space $H_{K(a,t)}$, the norm of $h(z)$ in $H_{k(a,t)}$ is expressible in the form

$$\|h\|^2_{H_{K(a,t)}} = \frac{1}{\sqrt{2\pi t(1-2at)}} \iint_{\mathbb{C}} \left| h(z) \exp\left\{ \frac{-az^2}{2(1-2at)} \right\} \right|^2 \cdot \exp\left\{ -\frac{y^2}{2t(1-2ar)} \right\} dx\, dy$$
$$= \sum_{n=0}^{\infty} \frac{\{2t(1-2at)\}^n}{n!} \int_{\mathbb{R}} \left| \partial_x^n \left[h(x) \exp\left\{ \frac{-ax^2}{2(1-2at)} \right\} \right] \right|^2 dx. \tag{5.2}$$

Hence, from the reproducing property in $H_{K(a,t)}$, we have

$$h(z) = (h(\cdot), K(\cdot, \bar{z}; a, t))_{H_{K(a,t)}}$$
$$= \sum_{n=0}^{\infty} \frac{\{2t(1-2at)\}^n}{n!} \int_{\mathbb{R}} \partial_x^n \left[h(\xi) \exp\left\{ \frac{1}{2} W_{a,t}(\xi) \right\} \right] \cdot \overline{\partial_\xi^n \left[K(\xi, \bar{z}; a, t) \exp\left\{ \frac{1}{2} W_{a,t}(\xi) \right\} \right]}\, d\xi.$$

6. Approximation by the Solutions

For any $h \in L^2[\mathbb{R}, \exp\{W_{a,t}(x)\}\, dx]$, we construct a sequence $\{u_n(z)\}$ such that $u_n(z) \in H_{K(a,t)}$ and

$$\lim_{n\to\infty} \int_{\mathbb{R}} |u_n(x) - h(x)|^2 \exp\{W_{a,t}(x)\}\, dx = 0.$$

First note that the images

$$f(z) = (T^*g)(z) = (g(\cdot), TK(\cdot, \bar{z}; a, t))_{L^2[\mathbb{R}, \exp\{W_{a,t}(x)\}\, dx]}$$

of the members g in $L^2[\mathbb{R}, \exp\{W_{a,t}(x)\}\, dx]$ are characterized as the reproducing kernel Hilbert space $H_{\mathbb{K}(a,t)}$ whose reproducing kernel is

$$\mathbb{K}(z, \bar{u}; a, t) = (TK(\cdot, \bar{u}; a, t), TK(\cdot, \bar{z}; a, t))_{L^2[\mathbb{R}, \exp\{W_{a,t}(x)\}\, dx]}.$$

From the concrete expression (4.4) of $\mathbb{K}(z, \bar{u}; a, t)$ we see that the family of functions

$$\varphi_n(z) = \frac{1}{2}\left[n!\{8t(1-2at)\}^n \sqrt{\pi t(1-2at)}\right]^{-1/2} \cdot \exp\left\{-\frac{(1-8at)z^2}{16t(1-2at)}\right\} z^n \qquad (n = 0, 1, 2, \dots) \quad (6.1)$$

is a complete orthonormal system in $H_{\mathbb{K}(a,t)}$. Note that

$$\varphi_n \in H_{k(a,t)}, \qquad n = 0, 1, 2, \dots .$$

We set

$$a_n = (T^*h, \varphi_n)_{H_{\mathbb{K}(a,t)}}$$

and

$$\hat{f}_N(z) = \sum_{n=0}^{N} a_n \varphi_n(z).$$

Then, $\hat{f}_N \in H_{k(a,t)} \subset H_{\mathbb{K}(a,t)}$ and the sequence $\{\hat{f}_N\}$ converges to T^*h in $H_{\mathbb{K}(a,t)}$ as N tends to infinity. As in the proof of Theorem 4.1, we construct the sequence u_N^* satisfying

$$T^*Tu_N^* = \hat{f}_N.$$

Since the adjoint operator T^* is an isometry of $L^2[\mathbb{R}, \exp\{W_{a,t}(x)\}\,dx]$ onto $H_{\mathbb{K}(a,t)}$, we obtain

THEOREM 5. *For any $h \in L^2[\mathbb{R}, \exp\{W_{a,t}(x)\}\,dx]$, let the sequence $\{u_N^*\}_{N=0}^{\infty}$ be constructed by the above method. Then, we have*

$$\lim_{N \to \infty} \int_{\mathbb{R}} |[Tu_N^*](x) - h(x)|^2 \exp\{W_{a,t}(x)\}\,dx = 0.$$

7. Representation of Passed Time

Most of initial value problems for the heat equations in which one is interested deal with initial values with compact supports. Meanwhile, we see, by some arguments in Section 2, that the solutions of them can be extended as entire functions for any fixed positive time point. Hence, the analyticity of such solutions may have a clue for representing the passed time in terms of the current heat distributions.

Although it is implicit, we obtain, by virtue of Theorem 3, a formulation as follows.

THEOREM 6. *Suppose that $h(x)$ ($\not\equiv 0$) is, to some fixed time $t_0 > 0$, the trace of the solution of an initial value problem for the heat equation in which the initial values are zero outside a bounded set and depend continuously on the space variable. Then the time t_0 is represented by*

$$t_0 = \sup\left\{t > 0; \iint_{\mathbb{C}} \left| \int_{-\infty}^{\infty} h(\xi) \exp\left(-\frac{\xi^2}{8t} + \frac{z\xi}{4t}\right) d\xi \right|^2 \cdot \exp\left(\frac{-3x^2 + y^2}{12t}\right) dx\, dy < \infty\right\}. \tag{7.1}$$

Proof. We denote the initial values by $F(x)$. Then we have the expression

$$h(x) = \frac{1}{\sqrt{4\pi t_0}} \int_{-\infty}^{\infty} F(\xi) \exp\left\{-\frac{(x-\xi)^2}{4t_0}\right\} dx.$$

Setting $a = 0$ in Theorem 3, it is sufficient to prove that the right hand side of (7.1) is finite, and that it is equal to or less than t_0. We assume that, for some $s > t_0$,

$$\iint_{\mathbb{C}} \left| \int_{-\infty}^{\infty} h(\xi) \exp\left(-\frac{\xi^2}{8s} + \frac{z\xi}{4s}\right) d\xi \right|^2 \exp\left(\frac{-3x^2 + y^2}{12s}\right) dx\, dy < \infty.$$

Again, by Theorem 3, there exists a function $G(x)$ in $L^2(\mathbb{R})$ such that

$$h(x) = \frac{1}{\sqrt{4\pi s}} \int_{-\infty}^{\infty} G(\xi) \exp\left\{-\frac{(x-\xi)^2}{4s}\right\} d\xi, \qquad x \in \mathbb{R},$$

and so we obtain, by the method of Fourier transform, the representation

$$F(x) = \frac{1}{\sqrt{4\pi(s-t_0)}} \int_{-\infty}^{\infty} G(\xi) \exp\left\{-\frac{(x-\xi)^2}{4(s-t_0)}\right\} d\xi, \qquad x \in \mathbb{R}.$$

This implies that F is the restriction of an entire function to $\mathbb{R}$. We therefore have a contradiction for our hypothesis.

Remark. In Theorem 2, we have, in general

THEOREM 7. *For $G \in L^p[\mathbb{R}, dx]$, $1 \leq p \leq \infty$, define a linear operator by*

$$[SG](x) = u_F(x,t)\exp\{\tfrac{1}{2}W_{a,t}(x)\},$$

where $F(x) = G(x)e^{ax^2/2}$. Then the following inequality holds:

$$\|SG\|_p \leq (1-2at)^{1/p-1/2}\|G\|_p, \qquad 1 \leq p \leq \infty.$$

Proof. For all $x \in \mathbb{R}$,

$$\begin{aligned}
|[SG](x)| &= \int_{\mathbb{R}} h(x-\xi,t)e^{a\xi^2/2}G(\xi)\,d\xi \exp\left\{\frac{1}{2}W_{a,t}(x)\right\} \\
&\leq \|G\|_\infty \int_{\mathbb{R}} h(x-\xi,t)e^{a\xi^2/2}\,d\xi \exp\left\{\frac{1}{2}W_{a,t}(x)\right\} \\
&= \frac{1}{\sqrt{1-2at}}\|G\|_\infty.
\end{aligned}$$

Also,

$$\begin{aligned}
\|SG\|_1 &\leq \int_{\mathbb{R}}\left(\int_{\mathbb{R}} h(x-\xi,t)|F(\xi)|\,d\xi\right)\exp\left\{\frac{1}{2}W_{a,t}(x)\right\}dx \\
&= \int_{\mathbb{R}}\left(\int_{\mathbb{R}} h(x-\xi,t)\exp\left\{\frac{1}{2}W_{a,t}(x)\right\}dx\right)|F(\xi)|\,d\xi \\
&= \sqrt{1-2at}\int_{\mathbb{R}}|F(\xi)|e^{-a\xi^2/2}\,d\xi \\
&= \sqrt{1-2at}\,\|G\|_1.
\end{aligned}$$

By the M. Riesz convexity theorem, our inequality is obtained.

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